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Double quantization of $\mathbf{CP}^n$ type orbits by generalized Verma modules

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Abstract

It is known that symmetric orbits in $\mathbf{g}^*$ for any simple Lie algebra $\mathbf{g}$ are equipped with a Poisson pencil generated by the Kirillov–Kostant–Souriau bracket and the reduced Sklyanin bracket associated to the "canonical" R-matrix. We realize quantization of the Poisson pencil on $\mathbf{CP}^n$ type orbits (i.e. orbits in $sl(n+1)^*$ whose real compact form is $\mathbf{CP}^n$) by means of q-deformed Verma modules.

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1. Introduction

The problem of quantization of Poisson brackets is one of the most important in mathematical physics. In the framework of the deformation quantization scheme going back to the works by A. Lichnerowicz and his school (cf. [3]) it can be formulated as follows. Given a variety M equipped with a Poisson bracket, it is necessary to construct a flat deformation $\mathcal{A}_\hbar$ of an algebra $\mathcal{A} = \mathrm{Fun}(M)$ of functions over M [1] such that the corresponding Poisson

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[1] We say that any algebra $\mathcal{A}_\hbar$ depending on a formal parameter $\hbar$ is a *flat deformation* (or simply, deformation) of $\mathcal{A}$ if $\mathcal{A} = \mathcal{A}_\hbar/\hbar\mathcal{A}_\hbar$ and $\mathcal{A}_\hbar$ is isomorphic to $\mathcal{A}[[\hbar]]$ as $\mathbf{C}[[\hbar]]$-modules. Hereafter, $V[[\hbar]]$ where V is a linear space stands for the completion of $V \otimes_{\mathbf{C}} \mathbf{C}[[\hbar]]$ in the $\hbar$-adic topology (in what follows the basic field is $k = \mathbf{C}$). Abusing notation we will let $\mathcal{A} \to \mathcal{A}_\hbar$ denote the deformation in question. Two-parameter flat deformation can be defined in a similar way.

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bracket (which exists for any flat deformation of a commutative algebra) coincides with the initial one.

The existence of such a quantization for any nowhere degenerated (i.e. defined by a symplectic structure) Poisson bracket had been shown in [5]. Recently, Kontsevich [28] has proved that any Poisson bracket is quantizable in the above sense.

Nevertheless, physicists are interested in an operator quantization, i.e. they want to realize the quantum algebra $\mathcal{A}_\hbar$ as an operator algebra in a linear (ideally, Hilbert) space. This enables them to carry out a spectral analysis of Himiltonians and to compute partition functions and other numerical characteristics of quantum models. Such a quantization of non-degenerated Poisson bracket (on any compact smooth variety) has been realized by Fedosov [14]. In fact, the famous Kirillov–Duflo orbit method which consists in assigning a representation $\rho: \mathbf{g} \to \operatorname{End} V$ of a Lie algebra $\mathbf{g}$ to an orbit $\mathcal{O} \subset \mathbf{g}^*$ can be considered as a particular case of the Fedosov approach. (We do not discuss here the limits of the orbit method, in the sequel we will restrict ourselves to semisimple orbits in $\mathbf{g}^*$ for simple Lie algebra $\mathbf{g}$.)

The quantization procedure suggested by Fedosov leads to an operator algebra equipped with a commutative trace. In fact, such a trace is delivered for appropriate quantum algebras by the Liouville measure of the initial Poisson bracket. However, a generic Poisson bracket does not possess any invariant measure and consequently it is not clear what is a trace in the corresponding quantum algebra.

In the early 1990s one of the author (DG) suggested certain Poisson brackets associated to classical R-matrices whose quantization leads to operator algebras in twisted categories. Traces in such algebras are also twisted (cf. [17,24]). These algebras arise from quantization of Poisson pencils generated by the linear Poisson–Lie bracket on $\mathbf{g}^*$ or its restriction to an orbit, called the Kirillov–Kostant–Souriau (KKS) bracket, and by a bracket naturally associated to a solution $R \in \wedge^2(\mathbf{g})$ of the classical *non-modified* Yang–Baxter equation.

$$[[R, R]] = [R^{12}, R^{13}] + [R^{12}, R^{23}] + [R^{13}, R^{23}] = 0 \tag{1.1}$$

Here, as usual $R^{12} = R \otimes \mathrm{id}$, etc.

Let us describe the latter bracket. Let M be a variety equipped with a representation $\rho: \mathbf{g} \to \operatorname{Vect}(M)$, where $\operatorname{Vect}(M)$ stands for the space of vector fields. Then the following bracket,

$$\{f, g\}_R = \mu\langle \rho^{\otimes 2}(R),\ \mathrm{d}f \otimes \mathrm{d}g\rangle, \quad f, g \in \operatorname{Fun}(M), \tag{1.2}$$

is Poisson. Here $\langle\ ,\ \rangle$ is the pairing between differential forms and vector fields on M extended to their tensor powers and μ is the usual commutative product in the space $\operatorname{Fun}(M)$. The bracket $\{\ ,\ \}_{\mathrm{R}}$ is called R-*matrix bracket.* If $M = \mathbf{g}^*$ or $M = \mathcal{O} \subset \mathbf{g}^*$ is an orbit we take ρ to be the coadjoint representation or its restriction to the orbit.

It is not difficult to see that in the last case any bracket of the family

$$\{\ ,\ \}_{a,b} = a\{\ ,\ \}_{\mathrm{KKS}} + b\{\ ,\ \}_{\mathrm{R}} \tag{1.3}$$

is Poisson. Here by $\{\ ,\ \}_{\mathrm{KKS}}$ we mean either the KKS bracket or the linear Poisson–Lie bracket on $\mathbf{g}^*$. Thus, we have a Poisson pencil well-defined on $\mathbf{g}^*$ or on an orbit in $\mathbf{g}^*$.

A procedure of quantizing this Poisson pencil can be realized in two steps (we call such procedures "double quantization"). In the first step one quantizes only the KKS bracket by means of the orbit method or by means of generalized Verma modules as it is described in Section 3. Then one twists the quantum operator algebra as it is described in Section 2. The resulting object is a two-parameter family of operator algebras in a twisted category. It comes with a deformed trace which is no longer commutative but it is S-commutative in the spirit of a super-trace. Here S is an involutive ($S^2 = \mathrm{id}$) twist, i.e. an operator acting in the tensor square of this algebra and satisfying the quantum Yang–Baxter equation (QYBE)

$$S^{12}S^{23}S^{12} = S^{23}S^{12}S^{23}.$$

Let us remark that by means of a similar twisting one can introduce natural "S-analogues" of basic objects of geometry and analysis. Thus, S-analogues of commutative algebras, vector fields, Lie algebras, (formal) Lie groups were defined, in the spirit of super-theory, in [15,18] (cf. also [22,24]).[2] However, a straightforward generalization of these notions to non-involutive twists (connected, say, to the quantum group (QG) $U_q(\mathbf{g})$) usually leads to a non-flat deformation. (From our viewpoint the principle "raison d'être" for objects belonging to the category of $U_g(\mathbf{g})$-modules is that they should represent a flat deformation of their classical counterparts.)

The main purpose of the paper is to realize an operator quantization of Poisson pencils (1.3) associated to the "canonical" classical R-matrix

$$R = \sum_{\alpha\in\Omega^+} \frac{X_\alpha \wedge X_{-\alpha}}{\langle X_\alpha, X_{-\alpha}\rangle} \in \wedge^2(\mathbf{g}), \tag{1.4}$$

where $\mathbf{g}$ is a complex simple Lie algebra, Ω^+ stands for the set of its positive roots with respect to a fixed triangular decomposition of $\mathbf{g}$ and $\langle\ ,\ \rangle$ stands for the Killing form.

This R-matrix satisfies the so-called classical *modified* YBE which means that the element $[[R, R]]$ is non-trivial and $\mathbf{g}$-invariant. Since this element is not identically zero, the associated R-matrix bracket is Poisson only on varieties where the three-vector field $\rho^{\otimes 3}([[R, R]])$ vanishes. Such varieties were called in [21] the *R-matrix type* varieties. All R-matrix type orbit in $\mathbf{g}^*$ were classified in [21]. In particular, all symmetric orbits in $\mathbf{g}^*$ are R-matrix type varieties. (Let us recall that an orbit $\mathcal{O}_x$ of a point x is called symmetric if there exists a decomposition $\mathbf{g} = \mathbf{k} \oplus \mathbf{m}$, where $\mathbf{k}$ is the stabilizer of x such that $[\mathbf{k}, \mathbf{k}] \subset \mathbf{k}$, $[\mathbf{m}, \mathbf{m}] \subset \mathbf{k}$, $[\mathbf{k}, \mathbf{m}] \subset \mathbf{m}$.)

Moreover, the R-matrix bracket over a symmetric orbit coincides with one of the two (left- or right-invariant) components of the Sklyanin bracket reduced to the orbit (recall that the Sklyanin bracket is the difference between the left- and the right-invariant brackets defined

[2] All these objects are also well-defined for some non-quasiclassical twists (i.e. twists which cannot be obtained by a deformation of the ordinary flip $S = \sigma$). One can naturally associate S-symmetric and S-skewsymmetric algebras of V to an involutive twist S: $V^{\otimes 2} \to V^{\otimes 2}$ where V is a linear space. If the twist is quasiclassical, i.e., if it is a deformation of the usual flip, then the Poincaré series of these algebras coincide with the classical ones, while those series corresponding to non-quasiclassical twists can be drastically different. The first examples of such twists were given in [15] (similar Hecke type twists were introduced in [16]).

by (1.2), where ρ is the natural homorphism of $\mathbf{g}$ into the space of left- or right-invariant vector fields on the corresponding group G). Meanwhile, the other component being reduced becomes proportional to the KKS bracket. This implies that on any symmetric orbit the R-matrix and the KKS brackets are compatible and therefore they generate the Poisson pencil (1.3).

Note that the one-sided invariant components of the Sklyanin bracket can be reduced to any semisimple (ss) orbit in $\mathbf{g}^*$ (i.e, to that of an ss element), but each of them becomes a Poisson bracket only on symmetric orbits (cf. [8,27]).

The Poisson pencil (1.3) with the R-matrix (1.4) on symmetric orbits has been quantized in the spirit of deformation quantization in [11]. The resulting object of the quantization procedure suggested in [11] is a two-parameter family of associative $U_g(\mathbf{g})$-invariant algebras. Let us make it precise that an algebra $\mathcal{A}$ is called $U_q(\mathbf{g})$-invariant (or $U_q(\mathbf{g})$-covariant) if

$$u \cdot (x_1 x_2) = (u_{(1)} \cdot x_1)(u_{(2)} \cdot x_2) \quad \forall u \in U_q(\mathbf{g}), \quad x_1, x_2 \in \mathcal{A}. \tag{1.5}$$

Hereafter $u_{(1)} \otimes u_{(2)}$ stands for $\Delta(u)$ (Sweedler's notations). The algebras $\mathcal{A}$ possessing this property will be called *quantum or braided* algebras, while by *twisted algebras* we mean the algebras belonging to a twisted category equipped with an arbitrary twist.

However, the product in the quantum algebra is realized in [11] by a series in two formal parameters, meanwhile the QG $U_q(\mathbf{g})$ appears as $U(\mathbf{g})[[\nu]]$ but equipped with a deformed coproduct (the so-called, Drinfeld's realization, see Section 2).

In the present paper we perform a double quantization of $\mathbf{CP}^n$ type orbits by the operator method. By $\mathbf{CP}^n$ type orbits we mean the orbits in $sl(n)^*$ of elements $\mu\omega_1$ or $\mu\omega_{n-1}$ where $\omega_1(\omega_{n-1})$ is the first (the last) fundamental weight of $sl(n)$ and $\mu \in \mathbf{C}$ is an arbitrary non-trivial factor. Compact forms of these complex orbits are just $\mathbf{CP}^{n-1}$ embedded into $su(n)^*$ as closed algebraic varieties.

More precisely, we represent our two-parameter quantum object $\mathcal{A}_{\hbar,q}$ as an operator algebra in braided (or q-deformed) generalized Verma modules. Similar to the previous case arising from the classical non-modified YBE, our quantization procedure consists of two steps.

The first, "classical", step is realized as follows. There exists a natural way to quantize ss orbits in $\mathbf{g}^*$ for any simple Lie algebra $\mathbf{g}$ by generalized Verma modules. Let M_ω be such a module of highest weight ω (its construction is given in Section 3) and $\rho_\omega\colon T(\mathbf{g}) \to \operatorname{End} M_\omega$ be the corresponding representation of the free tensor algebra $T(\mathbf{g})$. Then, the operator algebras $\mathcal{A}_\hbar = \operatorname{Im}\hbar\rho_{\omega/\hbar} \subset \operatorname{End}M_\omega[[\hbar]]$ can be treated as quantum objects with respect to the KKS bracket on the orbit $\mathcal{O}_\omega \subset \mathbf{g}^*$ of the element ω (we regard ω as an element of $\mathbf{g}^*$, see an explanation below). The passage from the representation ρ_ω to the representation $\hbar\rho_{\omega/\hbar}$ will be referred to as a renormalization procedure.

Let us remark that the operator algebra $\mathcal{A}_\hbar$ is an object of the category of $\mathbf{g}$-invariant algebras similar to the initial function algebra $\mathcal{A} = \operatorname{Fun}(\mathcal{O}_\omega)$.

The second step consists of a braiding of the algebras $\mathcal{A}_\hbar$. As a result we get the above-mentioned two-parameter family of $U_q(\mathbf{g})$-invariant operator algebras $\mathcal{A}_{\hbar,q}$. Let us emphasize that our approach to representing quantum algebras by means of braided generalized

Verma modules has the following advantage. The parameters $\hbar$ and q can be specialized: the operator ralization of the algebra $\mathcal{A}_{\hbar,q}$ is well-defined for any value of $\hbar$ and a generic q.

Moreover, the flatness of deformation $\mathcal{A} \to \mathcal{A}_{\hbar,q}$ is assured automatically. Let us remark that the quantum algebra $\mathcal{A}_{\hbar,q}$ can also be represented by a system of some algebraic equations. For the $\mathbf{CP}^n$ type orbits these equations are quadratic-linear-constant. It is not so difficult to guess a general form of these equations. The problem is to find the exact meaning of factors arising in them which ensure the flatness of deformation of the corresponding quotient algebras. Various ways to look at these factors were discussed in [8,9,19,23]. The "operator method" presented here is the most adequate way to solve this problem.

Thus, compared with [11] our approach enables us to realize quantum counterparts of the Poisson pencil in question explicitly in the spirit of non-commutative algebraic geometry.

The paper is organized as follows. In Section 2 we describe various algebraic structures connected to involutive twists arising from quantization of R-matrices satisfying the classical *non-modified* YBE. We show that certain quotients of twisted Hopf algebras are the appropriate objects allowing to explicitly describe the quantized orbits in $\mathbf{g}^*$. We also analyse the difference between this case (we refer to it as triangular or involutive) and the other one connected to the quasitriangular QG $U_q(\mathbf{g})$.

Section 3 is devoted to the "classical step" of quantization. The final object of this step is the mentioned above family of algebras $\mathcal{A}_\hbar$. Then we realize a q-deformation of these algebras as follows. We equip $\mathbf{g} = sl(n)$ with a structure of a $U_q(\mathbf{g})$-module, extend the action of the QG $U_q(\mathbf{g})$ to its enveloping algebra and represent this algebra in the q-deformed generalized Verma modules considered on the first step. These constructions are described in Sections 4 and 5. They result in a two-parameter family $\mathcal{A}_{\hbar,q}$ presented in Section 6.

Completing the introduction we want to pose the following question: how is it possible to define a proper trace in a quantum algebra arising from a given Poisson bracket by virtue of [28]? As our examples show, such traces are not necessarily commutative. (Although we are dealing with the complexification of $\mathbf{CP}^n$ the trace defined by a projection of the algebra $\mathcal{A}_{\hbar,q}$ to its trivial component is well defined on this algebra since it corresponds to the compact form of the orbits in question, cf. [25] where such a trace is studied in the $sl(2)$ case.) Thus, quantizing certain Poisson structures we should enlarge the framework of the ordinary Quantum Mechanics and use operator algebras belonging to twisted categories. We consider this approach as a further step in constructing generalized (or twisted) Quantum Mechanics, including Quantum super-Mechanics and that dealing with involutive twists suggested in [24]. We also hope that an extension of this approach to infinite-dimensional Lie algebras could be useful for understanding the quantum anomaly problem.

2. Triangular and quasitriangular cases: Comparative description

2.1

Let us first consider certain algebraic structures arising from R-matrices satisfying the classical YBE (1.1). Let us fix such an R-matrix R.

By a Drinfeld's result [12] there exists a series $F = F_\nu \in U(\mathbf{g})^{\otimes 2}[[\nu]]$ quantizing the R-matrix R in the following sense: $F_\nu = 1 + \nu P + \cdots$, where $P \in \mathbf{g}^{\otimes 2}$, $P - P^{21} = 2R$, and

$$\Delta^{12} F F^{12} = \Delta^{23} F F^{23}, \qquad \varepsilon^1 F = \varepsilon^2 F = 1. \tag{2.1}$$

Here $\Delta: U(\mathbf{g}) \to U(\mathbf{g})^{\otimes 2}$ is the usual coproduct and $\varepsilon : U(\mathbf{g}) \to \mathbf{C}$ is the counit in $U(\mathbf{g})$ (all operators are assumed to be naturally extended to $U(\mathbf{g})[[\nu]]$). Using F one can deform the usual Hopf structure of the algebra $U(\mathbf{g})$ in (at least) two different ways.

The first way consists of the following procedure. Let us introduce a new coproduct setting

$$\Delta_F(u) = F^{-1}\Delta(u)F = F^{-1}_{(1)} u_{(1)} F_{(1)} \otimes F^{(-1)}_{(2)} u_{(2)} F_{(2)}.$$

Here $F_{(1)} \otimes F_{(2)}$ (resp., $F^{-1}_{(1)} \otimes F^{-1}_{(2)}$ stands for F (resp., F^{-1}).

Then the algebra $U(\mathbf{g})[[\nu]]$ equipped with the initial product and unit, the coproduct Δ_F and the uniquely defined counit and antipode (cf. [13,20] where the antipode is expressed via F) becomes a Hopf algebra looking like the famous QG $U_q(\mathbf{g})$. Let us denote it by H.

Another way consists of a simultaneous deformation of the product and coproduct as follows:

$$\overline{\Delta}(u) = ad\ F^{-1}(\Delta(u)) = adF^{-1}_{(1)}(u_{(1)}) \otimes ad\ F^{-1}_{(2)}(u_{(2)})$$

and

$$\overline{\mu}(u_1 \otimes u_2) = \mu(ad\ F(u_1 \otimes u_2)) = \mu(adF_{(1)}(u_1) \otimes ad\ F_{(2)}(u_2)) \tag{2.2}$$

Here μ is the initial product in $U(\mathbf{g})$ and $ad\ F^{\pm 1}$ is defined by

$$adX(Y) = [X, Y]$$

and

$$ad\ (X_1 X_2 \ldots X_p)(Y) = adX_1(adX_2(\cdots adX_p(Y)\cdots)).$$

The space $U(\mathbf{g})[[\nu]]$ equipped with these product and coproduct, the classical unit, counit and antipode becomes a twisted Hopf algebra. Essentially, this means that

$$\overline{\Delta}\overline{\mu}(u_1 \otimes u_2) = (\overline{\mu} \otimes \overline{\mu})(\mathrm{id} \otimes S \otimes \mathrm{id})(\overline{\Delta}(u_1) \otimes \overline{\Delta}(u_2)), \tag{2.3}$$

where $S = S_\nu = F^{-1}\sigma F$ and σ is the flip (F and F^{-1} act in the above sense by $ad \otimes ad$). Let $\overline{H}$ denote this twisted Hopf algebra. The reader can easily verify that the operator S satisfies the QYBE.

Let us observe that the Δ-primitive elements $X \in U(\mathbf{g})$, i.e., such that $\Delta(X) = X \otimes 1 + 1 \otimes X$ (they are just the elements of the algebra $\mathbf{g}$) are still $\overline{\Delta}$-primitive. This follows from the second relation of (2.1).

The algebra $\overline{H}$ can be treated as the enveloping algebra of a generalized (or S-) Lie algebra defined by the deformed Lie bracket $[\ ,\ ]_\nu = [\ ,\ ]F_\nu$ or in more detailed form

$$[X, Y]_\nu = [adF_{(1)}(X), adF_{(2)}(Y)].$$

An axiomatic description of such type of brackets is given, for example, in [18]. Let $\mathbf{g}_\nu$ denote the space $\mathbf{g}[[\nu]]$ equipped with the bracket $[\ ,\ ]_\nu$. Its enveloping algebra defined naturally by

$$U(g_\nu) = T(g_\nu)[[\nu]]/\{x \otimes y - S(x \otimes y) - [x, y]_\nu\} \tag{2.4}$$

is filtrated quadratic (more precisely, the ideal is generated by quadratic-linear elements.)

Hereafter $T(V)$ stands for the free tensor algebra of a linear space V and $\{I\}$ stands for its ideal generated by a subset $I \subset T(V)$.

We also need the algebra $\mathcal{A}_{\hbar,\nu} = U(\mathbf{g}_\nu)_\hbar$ defined by formula (2.4) but with the bracket $[\ ,\ ]_\nu$ replaced by $\hbar[\ ,\ ]_\nu$. The algebra $\mathcal{A}_{\hbar,\nu}$ is also filtered quadratic and moreover, possesses a twisted Hopf structure. Moreover, we have by construction the following:

Theorem 1. *The two-parameter family $\mathcal{A}_{\hbar,\nu}$ is a flat deformation of the algebra $Sym(\mathbf{g}) = Fun(\mathbf{g}^*)$. The corresponding Poisson pencil is just* (1.3) *where $\{\ ,\ \}_{\mathrm{KKS}}$ is the linear extension of KKS bracket (Poisson–Lie one) and the bracket $\{\ ,\ \}_{\mathrm{R}}$ is associated to the initial R-matrix.*

By passing to the quotient $\mathcal{A}_\nu = \mathcal{A}_{\hbar,\nu}/\hbar\mathcal{A}_{\hbar,\nu}$ we get an S-commutative algebra which is also a flat deformation of the algebra $\mathrm{Fun}(\mathbf{g}^*)$. Let us make it precise that by this we mean an algebra $\mathcal{A} = \mathcal{A}_\nu$ equipped with an associative product $\mu: \mathcal{A}^{\otimes 2} \to \mathcal{A}$ and an involutive twist $S : \mathcal{A}^{\otimes 2} \to \mathcal{A}^{\otimes 2}$ such that $\mu S = \mu$ and $S\mu^{12} = \mu^{23}S^{12}S^{23}$. The last relation means that the product μ is S-invariant.

Now, let $\mathbf{g}$ be a simple Lie algebra. Then the enveloping algebra $\overline{H} = U(\mathbf{g}_\nu)$ is isomorphic to $U(\mathbf{g})[[\nu]]$. So, we can treat it as the algebra $U(\mathbf{g})[[\nu]]$ but equipped with a new coproduct (still denoted by $\overline{\Delta}$). Thus, we have equipped the algebra $U(\mathbf{g})[[\nu]]$ with two deformed coassociative coalgebraic structures converting it respectively into an Hopf algebra H and a twisted Hopf algebra $\overline{H}$.

However, in some sense the properties of the latter algebra are closer to those of the usual enveloping algebra $U(\mathbf{g})$. In the first place it is due to the fact that the algebra $\overline{H}$ possesses a generating set formed by $\overline{\Delta}$-primitive elements. Moreover, for this algebra its S-commutative analogue, i.e., the algebra $\mathcal{A}_\nu$, is well-defined and being equipped with the coproduct $\overline{\Delta}$ is still a twisted Hopf algebra, as in the classical case. The passage from the latter algebra to $\mathcal{A}_{\hbar,\nu}$ can be regarded as a twisted version of the quantization procedure of the linear Poisson–Lie bracket on $\mathbf{g}^*$ consisting in a passage from the symmetric algebra of $\mathbf{g}$ to the enveloping algebra $U(\mathbf{g})$.

By means of $\overline{\Delta}$-primitive elements it is not difficult to introduce the notion of twisted (or S-)vector fields: the twisted version of the Leibnitz rule for an involutive S is well-known. It is worth noticing that the twisted vector fields are just classical ones but their action on functions is deformed as follows:

$$\rho_\nu(X) \cdot a = \rho(adF_{(1)}(X)) \cdot \rho(F_{(2)})a, \quad X \in \mathbf{g}, \quad a \in \mathrm{Fun}(M), \tag{2.5}$$

where $\rho : \mathbf{g} \to \mathrm{Vect}(M)$ is a representation of $\mathbf{g}$ into the space of vector fields on a variety M extended to $U\mathbf{g}$.

Unfortunately, the Hopf algebra H does not have, in general, any generating set formed by Δ_F-primitive elements (the Δ-primitive elements are no longer Δ_F-primitive). This is a reason why it is not so clear what is the natural analogue of the Leibnitz rule related to the quantum group H (although some palliative forms of "quantum Leibnitz rule" can be sometimes suggested).

Let us consider now the category of $U(\mathbf{g})$-modules. It can be equipped with the twist

$$S_\nu^{U,V} = (\rho_V \otimes \rho_U)F^{-1}\sigma(\rho U \otimes \rho V)F : U \otimes V[[\nu]] \to V \otimes V[[\nu]],$$

where ρ_U is the representation of $U(\mathbf{g})$ in U. Thus, we have a twisted (tensor monoidal, in another terminology) category consisting of the same objects as the initial one but equipped with a new transposition.

This twisted category can be regarded as a category of H-modules and as a category of $\overline{H}$-modules. However, the action of an element $X \in \overline{H}$ on a tensor product of two modules U and V must be defined in spirit of formula (2.3) by means of the twist S_ν transposing $X_{(2)}$ and U (here $X_{(1)} \otimes X_{(2)} = \overline{\Delta}(X)$). In particular, in this way we can deform all (generalized) Verma modules into twisted ones.

Let us remark that the renormalization procedure mentioned in Section 1 (cf. also Section 3) has its twisted analogue. While in the classical case the map $\hbar\rho_{\omega/\hbar}$ sends $U(\mathbf{g})$ into End $M_\omega[[\hbar]]$ (i.e., the image does not contain negative powers of $\hbar$), in a deformed case such a property is satisfied only for an appropriated base in the deformed algebra. In the algebra $\overline{H}$ (which is isomorphic to H as an algebra) such a base is delivered by $\overline{\Delta}$-primitive elements.

Let us also mention the algebras dual to those H and $\overline{H}$. Both of them can be treated as deformations of the function algebra Fun(G) on the group G. However, if the former one looks like a famous "RTT=TTR" algebra and possesses a Hopf algebra structure, the latter one looks like a reflection equation (RE) algebra. For involutive twists it has been introduced in [15,18] under the name of a monoidal group. In more general setting RE algebras appear as dual objects of Majid's braided groups, cf. [31]. Majid has also suggested a transmutation procedure converting one algebra to another. RE algebras associated to twists depending on a spectral parameter were considered in [29].

2.2

Let us pass to the quasitriangular case, i.e., that related to the QG $U_q(\mathbf{g})$ where $\mathbf{g}$ is a complex simple Lie algebra. In this case there also exists a series F_ν quantizing the R-matrix (1.4) in the above sense. However, the first equation of (2.1) takes another form containing the Drinfeld's associator Φ (cf. [4]). Moreover, the corresponding twist takes the form

$$S = S_\nu = F^{-1}\sigma e^{\nu t/2}F, \tag{2.6}$$

where t is the split Casimir.

In this case the Hopf algebra H can be constructed in the same way as above. It is just the famous QG $U_q(\mathbf{g})$ but realized in an equivalent way as the algebra $U(\mathbf{g})[[\nu]]$ equipped with the deformed coproduct Δ_F (we call this form of the QG $U_q(\mathbf{g})$ its *Drinfeld's realization*). However, the above construction of the twisted algebra $\overline{H}$ is no longer valid because the product $\overline{\mu}$ defined as above is not associative (the associativity default is due to Drinfeld's associator).

Nevertheless, a twisted Hopf algebra arising from the QG H exists: it can be obtained from H by means of a transmutation procedure which is dual to that mentioned above. In fact, this procedure does not deform the algebraic structure and transforms the coproduct Δ_F into a new one $\overline{\Delta}$ converting the QG into a "braided group".

However, this braided Hopf algebra is rather useless for us since it does not apparently possess any base of $\overline{\Delta}$-primitive elements. In fact, instead of looking for an appropriate base in $U_q(\mathbf{g})$ we construct another, complementary, algebra which possesses such a base. More precisely, we will introduce a space $\mathbf{g}_q$ being nothing but $\mathbf{g}$ itself equipped with an action $U_q(\mathbf{g}) \to \mathrm{End}\, \mathbf{g}_q$ of the GQ and represent the tensor algebra $T(\mathbf{g}_q)$ into a q-deformed generalized Verma module M_ω^q with $\omega = \mu\omega_1$. Namely, the image of the algebra $T(\mathbf{g}_q)$ with μ expressed via $\hbar$ in a proper way provides us with the quantum counterpart of the Poisson pencil (1.3) on the $\mathbf{CP}^n$ type orbits (in the classical case $\hbar$ is proportional to μ^{-1} but in the quantum case their relation is a little bit more complicated). Hopefully, this method is valid for any symmetric orbit in $\mathbf{g}^*$ for any simple Lie algebra $\mathbf{g}$.

Note that although we do not embed the space $\mathbf{g}_q$ into the GQ $U_q(\mathbf{g})$, such an embedding exists in the $sl(n)$ case in virtue of Lyubashenko and Sudbery [30]. Using this embedding the authors of [30] have introduced a version of a quantum Lie $sl(n)$ bracket.

Completing this section we want to stress that in our approach the QG $U_q(\mathbf{g})$ play an auxiliary role. We use it only to describe the category to which our quantum algebras $\mathcal{A}_{\hbar,q}$ belong. Let us note that in the case when such a category is related to a non-quasiclassical twists mentioned in footnote 2 the algebras looking like $\mathcal{A}_{\hbar,q}$ can be constructed without any QG like objects. (In this case algebras of the "RTT=TTR" type can be introduced in the usual way, cf. [18], but their dual algebras differ drastically from the QG $U_q(\mathbf{g})$. We refer the reader to the paper [1] where an attempt to describe these algebras is undertaken.)

3. $\mathbf{CP}^n$ type orbits and their quantization by generalized Verma modules

Let us realize now the first, classical, step of the double quantization procedure for orbits in question.

Let $\mathbf{g}$ be a simple complex Lie algebra and let $\mathbf{g} = \mathbf{h} \oplus \mathbf{n}_+ \oplus \mathbf{n}_-$ be a fixed triangular decomposition where $\mathbf{h}$ is a Cartan subalgebra and $\mathbf{n}_\pm$ are nilpotent subalgebras. Consider a non-trivial element $\omega \in \mathbf{h}^*$ and extend it by 0 to the subalgebras $\mathbf{n}_\pm$. Thus, we can treat ω as an element of $\mathbf{g}^*$. Let $\mathcal{O}_\omega$ be the G-orbit of ω in $\mathbf{g}^*$ where G is the Lie group corresponding to $\mathbf{g}$ acting on $\mathbf{g}^*$ by coadjoint operators, and let

$$\{f, q\}_{\mathrm{KKS}}(x) = \langle [\mathrm{d}f, \mathrm{d}q], x\rangle, \quad x \in \mathcal{O}_\omega,$$

be the KKS bracket on $\mathcal{O}_\omega$.

It is well known that the orbit $\mathcal{O}_\omega$ is a closed algebraic variety in $\mathbf{g}^*$. Moreover, the space of (polynomial) functions $\mathcal{A} = \mathrm{Fun}\mathcal{O}_\omega)$ can be identified with the quotient $T(\mathbf{g})/\{I\}$, where I is some finite subset in $T(\mathbf{g})$

Thus, if $\mathcal{O}_\omega$ is a generic semisimple orbit (this means that in the decomposition $\omega = \sum \mu_i \omega_i$ where ω_i are fundamental weights $\mu_i \neq 0$ for any i) the family I consists of elements $x_i x_j - x_j x_i$, $1 \leq i, j \leq \dim \mathbf{g}$ and $C_i - c_i(\omega)$, $1 \leq i \leq \operatorname{rank} \mathbf{g}$ where C_i are invariant (Casimir) functions and $c_i(\omega)$ are certain constants depending on ω.

Let us consider another example of such type orbits, namely, those in $\mathbf{g}^* = sl(n)^*$ of elements $\omega = \mu\omega_1$ or $\omega = \mu\omega_{n-1}$ for some $\mu \in \mathbf{C}$. These orbits (or, more precisely, their real compact forms in $su(n)^*$) can be identified with $\mathbf{CP}^{n-1}$. They are called $\mathbf{CP}^n$ type orbits. It is well known that these orbits can be described by means of a system of quadratic equations. An explicit form of this system follows from the structure of $\mathbf{g}^{\otimes 2}$ as a $\mathbf{g}$-module. Let us perform such an analysis.

Proposition 1. *Let $\mathbf{g} = sl(n)$, $n \geq 4$. Then the highest weights of irreducible components of the $\mathbf{g}$-module $\mathbf{g}^{\otimes 2}$ are*

$$\begin{aligned} &2\omega_1 + 2\omega_{n-1}, \quad \omega_1 + \omega_{n-1}, \quad \omega_2 + 2\omega_{n-1}, \\ &2\omega_1 + \omega_{n-2}, \quad \omega_2 + \omega_{n-2}, \quad \text{and} \quad 0. \end{aligned} \tag{3.1}$$

All the irreducibles from (3.1) *occur in $\mathbf{g}^{\otimes 2}$ with multiplicity one except the irreducible with the highest weight $\omega_1 + \omega_{n-1}$ (being the highest weight of $\mathbf{g}$ itself) which occurs twice, once in the symmetric part I_+ of $\mathbf{g}^{\otimes 2}$ and once in the skewsymmetric part I_-.*

Note that in the $sl(2)$ case the decomposition (3.1) contains only the components with the highest weights.

$$0, \quad 2\omega_1, \quad 4\omega_1$$

all with multiplicity one and in the $sl(3)$ case the component of the highest weight $\omega_2 + \omega_{n-2}$ does not appear.

Let us denote the finite-dimensional irreducible $\mathbf{g}$-module with the highest weight λ by V_λ. The corresponding highest weight vector (assuming V_λ to be embedded in $\mathbf{g}^{\otimes 2}$) will be denoted by s_λ. For the highest weight $\omega_1 + \omega_{n-1}$ which occurs twice in (3.1) we denote $V^+_{\omega_1+\omega_{n-1}}$ (resp. $V^-_{\omega_1+\omega_{n-1}}$) the component of the highest weight $\omega_1 + \omega_{n-1}$ belonging to I_+ (resp. I_-). Their highest weight vectors will be denoted by $s^+_{\omega_1+\omega_{n-1}}$ (resp., $s^-_{\omega_1+\omega_{n-1}}$). The precise expressions for the corresponding highest weight vectors are presented in Proposition 3 with a specialization $q = 1$.

Then the orbit under consideration can be defined by the following system (here $n \geq 4$, the cases $n = 2, 3$ are left to the reader):

$$\begin{aligned} &V_{\omega_2+2\omega_{n-1}} = 0, \qquad V_{2\omega_1+\omega_{n-2}} = 0, \qquad V^-_{\omega_1+\omega_{n-1}} = 0 \\ &\quad (\text{or, equivalently, } x_i x_j - x_j x_i = 0 \,\forall i, j), \end{aligned} \tag{3.2}$$

$$V_{\omega_2+\omega_{n-2}} = 0, s_0 - c_0(\omega) = 0, \qquad V^+_{\omega_1+\omega_{n-1}} - c_1(\omega)\mathbf{g} = 0, \tag{3.3}$$

where $s_0 = C_1$ is a generator of the trivial module (Casimir element) and the constants $c_i(\omega)$, $i = 0, 1$, are

$$c_0(\mu\omega_1) = \frac{n-1}{n}\mu^2, \qquad c_1(\mu\omega_1) = 2\frac{n-2}{n}\mu \tag{3.4}$$

(if we normalize s_0, $s^1_{\omega_1+\omega_{n-1}}$ and $s^2_{\omega_1+\omega_{n-1}}$ by (4.15)–(4.17) with $q = 1$ and put $s^+_{\omega_1+\omega_{n-1}} = s^1_{\omega_1+\omega_{n-1}} + s^2_{\omega_1+\omega_{n-1}}$). The last equation of (3.3) is a symbolic form of the relation $s^+_{\omega_1+\omega_{n-1}} = c_1(\omega)g_{1,n}$ and all the descendants of this relation.

Thus, we have $\mathcal{A} = \mathrm{Fun}(\mathcal{O}_\omega) = T(\mathbf{g})/\{I\}$ with the family $I \subset \mathbf{C} \oplus \mathbf{g} \oplus \mathbf{g}^{\otimes 2}$ generated by the l.h.s. of formulae (3.2) and (3.3). So the algebra $\mathcal{A}$ is filtered quadratic. (Note that this system was given in [8] in a non-consistent form.)

Since the orbit $\mathcal{O}_\omega$ is a symmetric space it is a spherical or multiplicity free variety, i.e. in the decomposition of the space $\mathrm{Fun}(\mathcal{O}_\omega)$ into a direct sum of irreducibles their multiplicities are at most one.[3] It is well known that for the orbits of $\mathbf{CP}^n$ type

$$\mathrm{Fun}(\mathcal{O}_\omega) \approx \bigoplus_{k=0}^{\infty} V_{k(\omega_1+\omega_{n-1})}.$$

Let us now discuss a way to quantize the KKS bracket well-defined in the algebras $\mathcal{A} = \mathrm{Fun}(\mathcal{O}_\omega)$ by means of generalized Verma modules.

Let K be the stabilizer of the point $\omega \in \mathbf{g}^*$. So, $\mathcal{O}_\omega = G/K$. Let $\mathbf{k} = \mathrm{Lie}(K)$ be the Lie algebra of the group K and $\mathbf{p} = \mathbf{k} + \mathbf{n}_+$ be a parabolic subalgebra of $\mathbf{g}$. Let us consider the induced $\mathbf{g}$-module

$$M_\omega = Ind^{\mathbf{g}}_{\mathbf{p}} \mathbf{1}_\omega = U(\mathbf{g}) \otimes_{U(\mathbf{p})} \mathbf{1}_\omega,$$

where $\mathbf{1}_\omega$ is the one-dimensional $\mathbf{p}$-module equipped with the representation

$$\rho_\omega(x)e = \langle \omega, x\rangle e, \quad x \in \mathbf{p}$$

(e is a generator of the module). The $\mathbf{g}$-module M_ω is usually called a *generalized* (in the sequel we omit this precision) *Verma module*. Let $\rho_\omega : \mathbf{g} \to \mathrm{End} M_\omega$ denote the induced representation.

The operator algebra End M_ω is a quantum object with respect to the algebra of functions Fun $(\mathcal{O}_\omega)$. To give an exact meaning to this statement let us introduce an associative algebra $\mathcal{A}_\hbar$ depending on a parameter $\hbar$ as follows. Let us consider a map $\overline{\rho}_\hbar = \hbar\rho_{\omega/\hbar} : \mathbf{g} \to$ End $M_\omega[[\hbar]]$, extend it naturally to $T(\mathbf{g})$ and introduce the algebra $\mathcal{A}_\hbar$ as a subalgebra of End $M_\omega[[\hbar]]$ being, by definition, the image $\overline{\rho}_\hbar(T(\mathbf{g}))$.

Proposition 2. *The algebra $\mathcal{A}_\hbar$ is a flat deformation of $\mathcal{A} = \mathrm{Fun}(\mathcal{O}_\omega)$ and the corresponding Poisson bracket is just the KKS one.*

[3] Let us remark that the only symmetric orbits corresponding to the Lie algebra $\mathbf{g} = sl(n)$ are

$$\mathcal{O}_x = SL(n)/S(L(k) \times L(n-k)), \quad 1 \le k \le n-1,$$

whose real compact forms are Grassmanians (the cases $k = 1$ and $k = n-1$ correspond to $\mathbf{CP}^{n-1}$). Symmetric orbits in $\mathbf{g}^*$ for other simple Lie algebras $\mathbf{g}$ have been classified by E. Cartan (cf. [26,27]).

This statement is valid for any simple Lie algebra and for any ss orbit. We demonstrate it for the orbits of $\mathbf{CP}^n$ type where all the calculations can be easily done. In fact we will see that in this case the algebras $\mathcal{A}$ and $\mathcal{A}_\hbar/\hbar\mathcal{A}_\hbar$ are isomorphic as $\mathbf{g}$-modules (they are consisting of the same irreducibles with multiplicity one).

Let us first consider the finite-dimensional $sl(n)$-modules V_ω, $\omega = \mu\omega_1$, $\mu \in \mathbf{Z}_+$, where $\mathbf{Z}_+$ stands for the set of non-negative integers. Such a module can be naturally identified with a symmetric power of the vector fundamental space V_{ω_1} (in the $sl(2)$ case the factor $\frac{1}{2}\mu$ is just the spin of the module). Its dimension is equal to $\binom{\mu+n-1}{n-1}$.

Let us fix the following base in the space V_ω:

$$|m_1, \dots, m_n\rangle = x_1^{m_1}, \cdots, x_n^{m_n}, \quad \sum m_i = \mu. \tag{3.5}$$

Let $h_i \in \mathbf{h}, e_i \in \mathbf{n}_+, f_i \in \mathbf{n}_-, 1 \le i \le n-1$, be a standard Chevelley base in the Lie algebra $sl(n)$. The elements h_i, e_i, f_i act in the module V_ω as first order differential operators

$$e_i = x_i \frac{\partial}{\partial x_{i+1}}, \quad f_i = x_{i+1}\frac{\partial}{\partial x_i}, \quad h_i = x_i \frac{\partial}{\partial x_i} - x_{i+1}\frac{\partial}{\partial x_{i+1}}. \tag{3.6}$$

In the base (3.5) the operators (3.6) look like

$$\begin{aligned} e_i|m_1, \dots, m_n\rangle &= m_{i+1}|m_1, \dots, m_{i+1}, m_{i+1} - 1, \dots, m_n\rangle, \\ f_i|m_1, \dots, m_n\rangle &= m_i|m_1, \dots, m_i - 1, m_{i+1} + 1, \dots, m_n\rangle, \\ h_i|m_1, \dots, m_n\rangle &= (m_i - m_{i+1})|m_1, \dots, m_i, m_{i+1}, \dots, m_n\rangle. \end{aligned} \tag{3.7}$$

It is well known that the $sl(n)$-module $\mathrm{End}V_{\mu\omega_1}$, $\mu \in \mathbf{Z}_+$, is isomorphic to the following multiplicity free direct sum:

$$\mathrm{End}V_{\mu\omega_1} \approx \bigoplus_{k=0}^{\mu} V_{k(\omega_1+\omega_{n-1})}. \tag{3.8}$$

Now let us pass to the Verma module $M_\omega, \omega = \mu\omega_1, \mu \in \mathbf{C}$. Similar to the above finite-dimensional modules V_ω it possesses the following base:

$$|m_1 \dots, m_n\rangle, \quad \sum m_k = \mu, \quad m_k \in \mathbf{Z}_+, \quad k = 2, \dots, n. \tag{3.9}$$

Thus, the elements of the base (3.9) are labelled by the vectors $(m_2, \dots, m_n), m_k \in \mathbf{Z}_+$. The action of $sl(n)$ on $M_{\mu\omega_1}$ is given by formulae (3.6) and (3.7) as well.

Formula (3.8) must be modified as follows:

$$\mathrm{Im}\rho_\omega(T(\mathbf{g})) \approx \bigoplus_{k=0}^{\infty} V_{k(\omega_1+\omega_{n-1})}. \tag{3.10}$$

We have replaced End $V_{\mu\omega_1}$ by $\mathrm{Im}\rho_\omega(T(\mathbf{g}))$ since for infinite-dimensional modules the map ρ_ω is no longer surjective.

Let us now go back to the algebra $\mathcal{A}_\hbar = \mathrm{Im}\overline{\rho}_\hbar(T(\mathbf{g}) \subset \mathrm{End}M_\omega[[\hbar]]$. By means of the decomposition (3.10) it is easy to show that $\mathcal{A}_\hbar$ is a flat deformation of the algebra $\mathcal{A}$. The map $\overline{\rho}_\hbar$ sends the Chevalley generators into operators acting according to formulae (3.7) but with $m_2, \ldots, m_n$ replaced by $\hbar m_2, \ldots, \hbar m_n$. The commutators between the images of the Chevalley generators are those in $sl(n)$ multiplied by $\hbar$. This implies that the corresponding Poisson bracket is equal to the KKS one.

Now let us represent the algebra $\mathcal{A}_\hbar$ as a quotient

$$\mathcal{A}_\hbar = \mathrm{Im}\overline{\rho}_\hbar(T(\mathbf{g}) = T(\mathbf{g})[[\hbar]]/\mathrm{Ker}\,\overline{\rho}_\hbar.$$

In the case under consideration ($\mathbf{g} = sl(n), \omega = \mu\omega_1$) this quotient is also a quadratic algebra. More precisely, the ideal $\mathrm{Ker}\overline{\rho}_\hbar$ is generated by a finite family $I_\hbar \subset \mathbf{C} \oplus \mathbf{g} \oplus \mathbf{g}^{\otimes 2}$, looking like that defined by the l.h.s. of (3.2) and (3.3) but with some evident modifications: the elements $x_i x_j - x_j x_i$ must be replaced by those $x_i x_j - x_j x_i - \hbar[x_i, x_j]$ and the factors $c_i(\omega), i = 0, 1$, must be deformed to those $c_i(\omega, \hbar)$ depending on $\hbar$ (with $c(\omega, \hbar) = c(\omega) \mathrm{mod}\, \hbar$). Namely, with s_0 and $s^+_{\omega_1+\omega_{n-1}}$ normalized as in (3.4) we have

$$c_0(\mu\omega_1, \hbar) = \frac{n-1}{n}\mu(\mu + n\hbar), \qquad c_1(\mu\omega_1, \hbar) = \frac{n-2}{n}(2\mu + n\hbar).$$

4. Braided algebras

Our next aim is to braid the above quantization procedure. Let us begin with a description of the space $\mathbf{g}_q$ mentioned in Section 2.

Let $U_q(sl(n))$ be the quantum enveloping algebra corresponding to $sl(n)$. In the chevalley generators $e_i, f_i, h_i, i = 1, \ldots, n-1$, it could be described by the relations

$$[h_i, e_i] = 2e_i, \qquad [h_i, f_i] = -2f_i, \tag{4.1}$$

$$[h_i, e_{i\pm 1}] = -e_{-i\pm 1}, \qquad [h_i, f_{i\pm 1}] = f_{i\pm 1}, \tag{4.2}$$

$$[h_i, e_j] = [h_i, f_i] = 0, \quad |i - j| > 1, \tag{4.3}$$

$$[e_i, f_j] = \delta_{i,j}\frac{q^{h_i} - q^{-h_i}}{q - q^{-1}}, \tag{4.4}$$

$$e_i^2 e_{i\pm 1} - [2]_q e_i e_{i\pm 1} e_i + e_{i\pm 1} e_i^2 = 0, \tag{4.5}$$

$$f_i^2 f_{i\pm 1} - [2]_q f_i f_{i\pm 1} f_i + f_{i\pm 1} f_i^2 = 0, \tag{4.6}$$

with

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}} \quad \text{and} \quad q^{\alpha h_i} = \exp(\nu\alpha h_i).$$

We choose a comultiplication map as follows:

$$\Delta h_i = h_i \otimes 1 + 1 \otimes h_i, \qquad \Delta e_i = e_i \otimes 1 + q^{-h_i} \otimes e_i,$$
$$\Delta f_i = 1 \otimes f_i + f_i \otimes q^{h_i}. \tag{4.7}$$

Then the antipode has the form:

$$s(h_i) = -h_i, \qquad s(e_i) = -q^{h_i} e_i, \qquad s(f_i) = -f_i q^{-h_i}.$$

Let $\mathbf{g}_q$ be a q-analogue of the adjoint representation of the Lie algebra $sl(n)$ on itself, i.e., $\mathbf{g}_q$ is the (n^2-1)-dimensional $U_q(sl(n))$-module with the highest weight $\omega_1+\omega_{n-1}$. We want to explicitly describe the action of the QG $U_q(sl(n))$ on $\mathbf{g}_q$ in the fixed base of $\mathbf{g}_q$. Further we denote this action by $\mathrm{ad}=\mathrm{ad}_q$. Namely, the vector space $\mathbf{g}_q$ is generated by the elements $g_{i,j}, i, j=1,\ldots,n, i\neq j$ and $t_i, i=1,\ldots,n-1$. The action of Cartan elements coincides with the classical one:

$$\mathrm{ad}\, h_i(t_k)=0, \qquad \mathrm{ad}\, h_i(g_{k,l})=(\delta_{i,k}-\delta_{i,l}-\delta_{i+1,k}+\delta_{i+1,l})g_{k,l} \tag{4.8}$$

Non-trivial matrix coefficients of the action of the Chevalley generators e_i and f_i of $U_q(sl(n))$ look as follows:

$$\mathrm{ad}\, e_i(g_{a,i})=-g_{a,i+1}, \quad \mathrm{ad}\, e_i(g_{i+1,a})=g_{i,a}, \qquad a\neq i,i+1, \tag{4.9}$$

$$\mathrm{ad}, e_i(g_{i+1,i})=t_i, \quad \mathrm{ad}, e_i(t_i)=-[2]_q g_{i,i+1}, \quad \mathrm{ad}\, e_i(t_{i\pm 1})=g_{i,i+1},$$

$$\mathrm{ad}\, f_i(g_{a,i+1})=-g_{a,i}, \quad \mathrm{ad}\, f_i(g_{i,a})=g_{i+1,a}, \qquad a\neq i,i+1,$$

$$\begin{aligned}&\mathrm{ad}\, f_i(g_{i,i+1})=-t_i, \quad \mathrm{ad}\, f_i(t_i)=[2]_q g_{i+1,i},\\ &\mathrm{ad}\, f_i(t_{i\pm 1})=-g_{i+1,i}.\end{aligned} \tag{4.10}$$

So, we get matrix coefficients of this action from the classical ones replacing the coefficient 2 by its q-analogue $[2]_q=q+q^{-1}$.

Since $\mathbf{g}_q$ is a $U_q(sl(n))$-module, the tensor algebra $T(\mathbf{g}_q)$ can be equipped with a $U_q(sl(n))$-invariant product in the sense of (1.5). In what follows the algebra $T(\mathbf{g}_q)$ and all its $U_q(\mathbf{g})$-invariant quotients will be called *braided.*

In fact, the braided algebra $T(\mathbf{g}_q)$ is "too big" for us. We are really interested in its quotient over the kernel of map sending this algebra into End M^q_ω where M^q_ω is a q-analogue of the above Verma modules with $\omega=\mu\omega_1$. Namely this quotient with μ properly expressed via the parameter $\hbar$ plays the role of our "double quantum" object $\mathcal{A}_{\hbar,q}$.

Let us describe the mentioned kernel. To do this we need a decomposition of the $U_q(sl(n))$-module $\mathbf{g}_q^{\otimes 2}$ into a direct sum of irreducibles.

Proposition 3. *The formulae below describe all highest weight vectors of reducibles in the $U_q(sl(n))$-module $\mathbf{g}_q^{\otimes 2}(n\geq 4)$:*

$$s_{2\omega_1+2\omega_{n-1}}=g_{1,n}\otimes g_{1,n}, \tag{4.11}$$

$$s_{2\omega_1+\omega_{n-2}}=g_{1,n}\otimes g_{1,n-1}-q^{-1}g_{1,n-1}\otimes g_{1,n}, \tag{4.12}$$

$$s_{\omega_2+2\omega_{n-1}}=g_{1,n}\otimes g_{2,n}-q^{-1}g_{2,n}\otimes g_{1,n}, \tag{4.13}$$

$$\begin{aligned}s_{\omega_2+\omega_{n-2}}=&\,q g_{1,n}\otimes g_{2,n-1}+q^{-1}g_{2,n-1}\otimes g_{1,n}\\ &-g_{1,n-1}\otimes g_{2,n}-g_{2,n}\otimes g_{1,n-1}\end{aligned} \tag{4.14}$$

$$s^1_{\omega_1+\omega_{n-1}}=g_{1,2}\otimes g_{2,n}+q g_{1,3}\otimes g_{3,n}+\cdots+q^{n-3}g_{1,n-1}\otimes g_{n-1,n}$$

$$+q^{-2}\sum_{k=1}^{n-1}\frac{[n-k]_q}{[n]_q}t_k\otimes g_{1,n}-q^{n-2}\sum_{k=1}^{n-1}g_{1,n}\otimes\frac{[k]_q}{[n]_q}t_k \tag{4.15}$$

$$s^2_{\omega_1+\omega_{n-1}}=g_{2,n}\otimes g_{1,2}+q^{-1}g_{3,n}\times g_{1,3}+\cdots+q^{-n+3}g_{n-1,n}\otimes g_{1,n-1},$$

$$-q^{1-n}\sum_{k=1}^{n-1}\frac{[k]_q}{[n]_q}t_k\otimes g_{1,n}+q\sum_{k=1}^{n-1}g_{1,n}\otimes\frac{[n-k]_q}{[n]_q}t_k, \tag{4.16}$$

$$s_0=\sum_{i,j=1,i\le j}^{n-1}\frac{[i]_q[n-j]_q}{[n]_q}t_i\otimes t_j+\sum_{i,j=1,i>j}^{n-1}\frac{[j]_q[n-i]_q}{[n]_q}t_i\otimes t_j$$

$$+q\sum_{i<j}q^{j-i}g_{i,j}\otimes g_{j,i}+q^{-1}\sum_{i>j}q^{j-i}g_{i,j}\otimes g_{j,i}. \tag{4.17}$$

Because of the multiplicity of the highest weight $\omega_1+\omega_{n-1}$ component in this decomposition it is not so clear what are natural q-analogues $I^q_\pm$ of the symmetric I_+ and the skewsymmetric I_- components in $\mathbf{g}_q^{\otimes 2}$ (except of the $sl(2)$ case). [4]

Let us consider a way to introduce a decomposition $\mathbf{g}_q^{\otimes 2}=I^q_+\oplus I^q_-$ arising from an operator $\tilde{S}$ discussed in [8,11]. In Drinfeld's realization of the $QGU_q(\mathbf{g})$ the operator $\tilde{S}$ is defined by formula (2.6) but without the factor $e^{\nu t/2}$. So, it is evident that this operator is involutive. Moreover, being restricted to $\mathbf{g}_q^{\otimes 2}$ it has the same eigenspaces as the YB operator S but with eigenvalues ± 1. Namely, to pass from S to $\tilde{S}$ we must replace the eigenvalues of S close to 1 (resp., to -1) by 1 (resp., -1) assuming that $|q-1|\ll 1$.

We complete this section with describing the action of $\tilde{S}$ on the isotypical component of the highest weight $\omega_1+\omega_{n-1}$ (its action on other components of $\mathbf{g}_q^{\otimes 2}$ contains no new infromation for us). We use this computation in Section 6.

To do this we need a partial information on the quantum universal R-matrix in the $sl(n)$ case.

It is well known that the universal R-matrix $\mathcal{R}$ for the algebra $U_q(\mathbf{g})$ can be presented by

$$\mathcal{R}=\mathcal{R}_0\cdot q^{\sum c_{i,j}h_i\otimes h_j}, \tag{4.18}$$

where $(c_{i,j})$ is the matrix inverse to the Cartan matrix of $\mathbf{g}$ and $\mathcal{R}_0$ belongs to the tensor product of quantized enveloping algebras of nilpotent subalgebras $\mathbf{n}_\pm$ of $\mathbf{g}$: $\mathcal{R}_0\in U_q(\mathbf{n}_+)\otimes U_q(\mathbf{n}_-)$. Moreover, $\mathcal{R}_0=1 \bmod \mathbf{n}_+U_q(\mathbf{n}_+)\otimes U_q(\mathbf{n}_-)$.

In the $sl(n)$ case formula (4.18) has an especially simple form after embedding of $\mathcal{R}$ into $U_q(gl_n)\otimes U_q(gl_n)$:

$$\mathcal{R}=\mathcal{R}_0\cdot q^{\sum_{i=1}^n\varepsilon_i\otimes\varepsilon_i-1/n\left(\sum_{i=1}^n\varepsilon_i\right)\otimes\left(\sum_{i=1}^n\varepsilon_i\right)}, \tag{4.19}$$

where $h_i=\varepsilon_i-\varepsilon_{i+1}$, $i=1,\ldots,n-1$.

[4] For other simple Lie algebras $\mathbf{g}$ such natural q-analogues $I^q_\pm$ exist since $\mathbf{g}^{\otimes 2}$ is multiplicity free as a $\mathbf{g}$-module but the deformations $T(\mathbf{g})/\{I_\pm\}\to T(\mathbf{g}_q)/\{I^q_\pm\}$ are not flat. In the $sl(n)$ case one can split $\mathbf{g}_q^{\otimes 2}$ into a direct sum of components $I^q_\pm$ in such a way that these deformations are flat. It was shown in [6] by means of an embedding $\mathbf{g}_q\to U_q(\mathbf{g})$ which is slightly different from that considered in [30].

Let us compute the expressions $S(S^i_{\omega_i+\omega_{n-1}})$, $i = 1, 2$, where $S = \sigma(\mathrm{ad} \otimes \mathrm{ad})\mathcal{R}$ is the image of the universal R-matrix in the tensor square of the adjoint representation multiplied by the flip. The commutativity of S amd $\Delta(x)$ implies that $S(S^i_{\omega_1+\omega_{n-1}}) = \sum_j a_{i,j} S^j_{\omega_1+\omega_{n-1}}$, $i, j = 1, 2$, for some constants $a_{i,j}$.

The space $\mathbf{g}_q$ can be decomposed into three parts:

$$\mathbf{g_q} = g_+\mathbf{t} + \mathbf{g}_-,$$

where $\mathbf{g}_+$ is generated by the vectors $g_{i,j}$, $i < j$; $\mathbf{g}_-$ is generated by the vectors $g_{i,j}$, $i > j$; and $\mathbf{t}$ is generated by the elements t_i. Their crucial properties are

$$U_q(\mathbf{n}_\pm)\mathbf{t} \subset \mathbf{g}_\pm \quad \text{and} \quad U_q(\mathbf{n}_\pm)\mathbf{g}_\pm \subset \mathbf{g}_\pm.$$

One can observe from the explicit expressions for $S^i_{\omega_1+\omega_{n-1}}$ that

$$S^1_{\omega_1+\omega_{n-1}} = q^{-2}\sum_{k=1}^{n-1} \frac{[n-k]_q}{[n]_q} t_k \otimes g_{1,n} + \sum x_i \otimes y_i, \quad x_i \in \mathbf{g}_+, \tag{4.20}$$

$$S^2_{\omega_1+\omega_{n-1}} = q\sum_{k=1}^{n-1} \frac{[n-k]_q}{[n]_q} g_{1,n} \otimes t_k + \sum u_i \otimes v_i, \quad v_i \in \mathbf{g}_+. \tag{4.21}$$

Due to (4.19) we have

$$S(S^1_{\omega_1+\omega_{n-1}}) = q^{-2}\sum_{k=1}^{n-1} \frac{[n-k]_q}{[n]_q} t_k \otimes g_{1,n} + \sum x'_i \otimes y'_i, \quad x'_i \in \mathbf{g}_+,$$

which means, by virtue of (4.21), that

$$S(S^1_{\omega_1+\omega_{n-1}}) = q^{-3} S^2_{\omega_1+\omega_{n-1}}. \tag{4.22}$$

Analogously,

$$S(S^2_{\omega_1+\omega_{n-1}}) = q^{3-2n} S^1_{\omega_1+\omega_{n-1}}. \tag{4.23}$$

Formulae (4.22) and (4.23) show that the operator S is diagonal on the isotypical component $V^q_{\omega_1+\omega_{n-1}} \oplus V_{\omega_1+\omega^q_{n-1}}$ (as well as in the whole space $\mathbf{g}_q^{\otimes 2}$), has eigenvalues $\pm q^{-n}$ there, and the corresponding eigenvectors are

$$S_\pm = q^{2-n} S^1_{\omega_1+\omega_{n-1}} \pm q^{-1} S^2_{\omega_1+\omega_{n-1}}. \tag{4.24}$$

This result is true for $n \geq 3$. The case $n = 2$ is left for the reader (here $S_\pm = 0$).

5. Braided modules

Definition 1. *We say that M is a braided $T(\mathbf{g}_q)$-module (or, simply a braided module), if M is equipped with the structures of $U_q(sl(n))$-module and of $T(\mathbf{g}_q)$-module, and these structures are related as*

$$u \cdot (gm) = (u^{(1)} \cdot g)(u^{(2)} \cdot m) \tag{5.1}$$

for any $u \in U_q(sl(n))$, $g \in T(\mathbf{g}_q)$ *and* $m \in M$.

The braided algebra $T(\mathbf{g}_q)$ together with the category of its braided representations can be described also in the language of intertwining operators. Let M be a $U_q(sl(n))$-module. Then, by definition, the (second type) intertwining operator $\Psi^{\mathbf{g}_q}$ is a $U_q(sl(n))$-morphism

$$\Psi^{\mathbf{g}_q} : \mathbf{g}_q \otimes M \to M. \tag{5.2}$$

The components $\Psi_a^{g_q} : M \to M$ are defined via fixing the base $\mathbf{g}_a$ in $\mathbf{g}_q$:

$$\Psi_a^{\mathbf{g}_q}(m) = \Psi^{\mathbf{g}_q}(\mathbf{g}_a \otimes m). \tag{5.3}$$

Thus, M is a braided $T(\mathbf{g}_q)$-module if and only if there exists an action of the intertwining operator $\Psi^{\mathbf{g}_q}$ on M in the above sense.

Our next goal is to perform an explicit construction of certain braided modules. More precisely, we will define a $U_q(sl(n))$-morphism

$$T(\mathbf{g}_q) \to \mathrm{End}\, V_\omega^q,\ \omega = \mu\omega_1, \quad \mu \in \mathbf{Z}_+, \tag{5.4}$$

where V_ω^q is a q-deformed finite-dimensional module. After that we will extend this construction to the q-deformed Verma modules.

Since the module End V_ω is multiplicitly free, the component isomorphic to $\mathbf{g}$ is represented in it only once. The same is true for the $U_q(sl(n))$-module V_ω^q. This enables us to define the map (5.4) in a unique way, up to a factor, assuming it to be a $U_q(sl(n))$-morphism. Moreover, the space End V_ω does not contain any component with the highest weights $\omega_2 + 2\omega_{n-1}$ and $2\omega_1 + \omega_{n-2}$ (it is also true in a q-deformed case).

The $U_q(\mathbf{g})$-modules possessing these two properties were called braided in [19], here we use this term in a more general sense.

Let us now describe the map (5.4) explicitly. Since the algebra $T(\mathbf{g}_q)$ is generated by the space $\mathbf{g}_q$ and $U_q(sl(n))$ is generated by the Chevalley base, it suffices to ensure relation (5.1) for $u = h_i, e_i, f_i$ and for $g \in \mathbf{g}_q$. Below we write down these relations using the following traditional notation. Let M be a $U_q(sl(n))$-module and let $x \in M$ be an eigenvector of the action of the Cartan subalgebra $\mathbf{h}$ of $U_q(sl(n))$. Then we denote its eigenvalue by $\lambda(x) \in \mathbf{h}^*$, so that

$$h_i(x) = (\varepsilon_i - \varepsilon_{i+1}, \lambda(x)) \cdot x,$$
$$\varepsilon_i = \mathrm{diag}(0, \ldots, 0, 1, \ldots, 0),\ 1 \quad \text{at } i\text{th place.}$$

For instance, the weights $\lambda(g)$ of the representation ad in the space $\mathbf{g}_q$ coincide with the classical ones

$$\lambda(g_{i,j}) = \varepsilon_i - \varepsilon_j, \quad \lambda(t_i) = 0$$

and the action of the Cartan elements in g_q is given by the relation $\mathrm{ad} h_i(g) = (\varepsilon_i - \varepsilon_{i+1}, \lambda(g)) \cdot g$.

Proposition 4. *Let M be a finite-dimensional $U_q(sl(n))$-module. Then M is a braided $T(\mathbf{g}_q)$-module if and only if for any $g \in \mathbf{g}_q$ the following relations hold:*

$$[h_i, j] = \operatorname{ad} h_i(g), \tag{5.5}$$

$$[e_i, g]_{q^{-(\varepsilon_i-\varepsilon_{i+1},\lambda(g))}} = \operatorname{ad} e_i(g), \tag{5.6}$$

$$[f_i, g] = \operatorname{ad} f_i(g) \cdot q^{h_i}. \tag{5.7}$$

Here

$$[a, b]_q = ab - qba$$

and all the brackets are understood in the operator sense.

Proof. It suffices to say that for any finite-dimensional $U_q(sl(n))$-module M we treat

$$\mathrm{End} M = M \otimes M^*$$

as a left $U_q(sl(n))$-module where the action of $U_q(sl(n))$ on M^* is defined by means of the antipode s:

$$(v, u \cdot \xi) = (s(u) \cdot v, \xi),$$
$$v \in M, \quad \xi \in M^*, \quad u \in U_q(sl(n)).$$

The rest is a substitution of (4.7) in (5.1).

Let $V^q_{\omega_1}$ be the first (vector) fundamental representation of the algebra $U_q(sl(n))$. Similarly to the classical case let us consider the irreducible finite-dimensional representations $V^q_{\mu\omega_1}$, $\mu \in \mathbf{Z}_+$, of $U_q(sl(n))$ with the highest weights $\mu\omega_1$ in what follows we will omit q).

One can easily check that the operators $e_i, f_i, h_i \in \mathrm{End}\, V_{\mu\omega_1}$, whose non-trivial matrix elements are described in (5.8) satisfy relations (4.1)–(4.6) and thus define an action of the algebra $U_q(sl(n))$ in the vector space $V_{\mu\omega_1}$:

$$\begin{aligned} e_i|m_1, \ldots, m_n\rangle &= [m_{i+1}]_q |m_1, \ldots, m_i + 1, m_{i+1} - 1, \ldots, m_n\rangle, \\ f_i|m_1, \ldots, m_n\rangle &= [m_i]_q |m_1, \ldots, m_i - 1, m_{i+1} + 1, \ldots, m_n\rangle, \\ h_i|m_1, \ldots, m_n\rangle &= (m_i - m_{i+1})|m_1, \ldots, m_i, m_{i+1}, \ldots, m_n\rangle. \quad \square \end{aligned} \tag{5.8}$$

Proposition 5. *There is a unique structure (up to a multiplicative constant $\alpha \in \mathbf{C}$) of a braided $T(\mathbf{g}_q)$-module on the $U_q(sl(n))$-module $V_{\mu\omega_1}$.*

The action of generators of $T(\mathbf{g}_q)$ in the braided module $V_{\mu\omega_1}$ is:

$$\begin{aligned} &g_{i,j}|m_1, \ldots, m_n\rangle \\ &\quad = \alpha(\mu) q^{j+(m_1+\cdots+m_i)-(m_j+\cdots+m_n)} \\ &\qquad \times [m_j]_q |m_1, \ldots, m_i + 1, \ldots, m_j - 1, \ldots, m_n\rangle \quad \text{for } i < j \end{aligned} \tag{5.9}$$

$$g_{j,i}|m_1,\dots,m_n\rangle$$
$$= \alpha(\mu)q^{i-1+(m_1+\cdots+m_{i-1})-(m_{j+1}+\cdots+m_n)}$$
$$\times [m_i]_q|m_1,\dots,m_i-1,\dots,m_j+1,\dots,m_n\rangle \quad \text{for } i<j \tag{5.10}$$
$$t_i|m_1,\dots,m_n\rangle$$
$$= \alpha(\mu)q^{i+(m_1+\cdots+m_{i-1})-(m_{i+2}+\cdots+m_n)}$$
$$\times \frac{([2]_q q^{m_i-m_{i+1}} - q^{m_i+m_{i+1}+1} - q^{-m_i-m_{i+1}-1})}{q-q^{-1}}|m_1,\dots,m_n\rangle. \tag{5.11}$$

Proof. The proof goes by induction on the rank n. Let us start from the $U_q(sl(2))$ case. In this case relation (5.7) implies that

$$[g_{2,1}, f_1] = 0. \tag{5.12}$$

From the commutation relation (5.5) of $g_{2,1}$ with a Cartan element we see that in addition $g_{2,1}$ has the same matrix structure as f_1 and thus these two operators are proportional to each other. Applying relation (5.6) twice we get a description of the operators t_1 and $g_{1,2}$. Finally, we check that relations (5.5)–(5.7) are satisfied.

The passage to $U_q(sl(3))$ looks as follows. We know from the $sl(2)$ case that the operator $g_{2,1}$ has the form

$$g_{2,1} = \alpha(m_3)f_1. \tag{5.13}$$

We wish to find the normalization constant $\alpha(m_3)$. Applying the following particular cases of (5.6) and of (5.7):

$$g_{3,1} = [f_2, g_{2,1}]q^{-h_2}, \qquad g_{3,2} = -[e_1, g_{3,1}]$$

to the ansatz (5.13), we get a description of the operator $g_{3,2}$, depending on the choice of $\alpha(m_3)$. But we know again that

$$g_{3,2} = \alpha(m_1)f_2.$$

This gives a reccurence equation on $\alpha(m_3)$ which unique (up to constant factor) solution is $\alpha(m_3) = q^{-m_3}$. Then from (5.7) we get a description of other generators of $\mathbf{g}_q$ in the $sl(3)$ case. The general induction step is similar.

Now let us pass to q-deformed Verma modules. Let $M_\omega = M_\omega^q$ be such a module. It also possesses a base labelled by $(m_2,\dots,m_n)$, $m_i \in \mathbf{Z}_+$. For any $\mu \in \mathbf{C}$ there exists a $U_q(sl(n))$-invariant map

$$\bar{\rho}_\omega : T(\mathbf{g}_q) \to \mathrm{End}M_\omega, \quad \omega = \mu\omega_1. \tag{5.14}$$

This map is also defined (uniquely up to a factor) by formulae (5.9)–(5.11).

We are interested in the ideal Ker $\bar{\rho}_\omega$. It is also generated by its quadratic part $I_q(\mu,\alpha) = \mathrm{Ker}\bar{\rho}_\omega \cap (\mathbf{C} \oplus \mathbf{g}_q \oplus \mathbf{g}_q^{\otimes 2})$. To describe this quadratic part we consider the images of the highest weight elements S_0, $S^2_{\omega_1+\omega_{n-1}}$ in $g_q^{\otimes 2}$ with respect to the map $\bar{\rho}_\omega$.

The operator s_0 is scalar (we omit the symbol $\bar{\rho_\omega}$):

$$s_0 = \alpha(\mu)^2 q^n \frac{[n-1]_q}{[n]_q}[\mu]_q[\mu+n]_q Id, \tag{5.15}$$

and the operators $s^1_{\omega_1+\omega_{n-1}}$ and $s^2_{\omega_1+\omega_{n-1}}$ are proportional to the operator $g_{1,n}$:

$$s^1_{\omega_1+\omega_{n-1}} = \alpha(\mu)q^{n-2}\frac{[n-1]_q[\mu+n]_q-[\mu]_q}{[n]_q}g_{1,n}, \tag{5.16}$$

$$s^2_{\omega_1+\omega_{n-1}} = \alpha(\mu)q\frac{[n-1]_q[\mu]_q-[\mu+n]_q}{[n]_q}g_{1,n}. \tag{5.17}$$

Finally, we have the following description of $I_q(\mu,\alpha)$. □

Proposition 6. *The subspace $I_q(\mu,\alpha) \subset (\mathbf{C}\oplus\mathbf{g}_q\oplus\mathbf{g}_q^{\otimes 2})$ is a $U_q(sl(n))$-module generated by*

(i) *the highest weight vectors $s_{2\omega_1+\omega_{n-2}}$, $s_{\omega_2+2\omega_{n-1}}$, $s_{\omega_2+\omega_{n-2}}$,*

(ii) *the following combinations of highest weight vectors:*

$$s^1_{\omega_1+\omega_{n-1}} - \alpha(\mu)q^{n-2}\frac{[n-1]_q[\mu+n]_q-[\mu]_q}{[n]_q}g_{1,n}, \tag{5.18}$$

$$s^2_{\omega_1+\omega_{n-1}} - \alpha(\mu)q\frac{[n-1]_q[\mu]_q-[\mu+n]_q}{[n]_q}g_{1,n}, \tag{5.19}$$

$$s_0 - \alpha(\mu)^2 q^n \frac{[n-1]_q}{[n]_q}[\mu]_q[\mu+n]_q \cdot 1. \tag{5.20}$$

Therefore for the elements $s_\pm$ defined by (4.24) we have the following formula:

$$s_\pm = \alpha(\mu)\frac{[n-1]_q \pm 1}{[n]_q}([\mu+n]_q \pm [\mu]_q)g_{1,n}. \tag{5.21}$$

6. The algebra $\mathcal{A}_{\hbar,q}$ and quantum $\mathbf{CP}^n$ type orbits

Let us define now the two-parameter algebra $\mathcal{A}_{\hbar,q}$ using the results of the previous sections. To do this we must express μ via $\hbar$ and choose the factor $\alpha(\mu)$ in proper way. In the classical case ($q=1$) by setting $\hbar = \mu^{-1}, \alpha(\mu) = \alpha_0\hbar$ we get an algebra which differs from the above algebra $\mathcal{A}_\hbar$ by a renormalization of the parameter. Thus, the algebra $\mathcal{A}_\hbar/\hbar\mathcal{A}_\hbar$ is just the function algebra on the corresponding orbit (labelled by α_0).

In the quantum case ($q \neq 1$) we suppose that $|g| \neq 1$. This condition is motivated by our desire ti have $[\mu]_q \to \infty$ as $\mu \to \infty$.

Let us set

$$\alpha(\mu) = \frac{\alpha_0}{[\mu]_q} \quad \text{and} \quad \frac{[\mu+n]}{[\mu]_q} = \gamma(q) + \hbar, \tag{6.1}$$

where $\gamma(q) = q^n$ if $|q| > 1$ and $\gamma(q) = q^{-n}$ if $|q| < 1$. We use (6.1) as the definition of the parameter $\hbar$.

Then the elements (5.18)–(5.20) become

$$s^1_{\omega_1+\omega_{n-1}} - \alpha_0 q^{n-2} \frac{[n-1]_q(\gamma(q)+\hbar)-1}{[n]_q} g_{1,n}, \tag{6.2}$$

$$s^2_{\omega_1+\omega_{n-1}} - \alpha_0 q \frac{[n-1]_q-(\gamma(q)+\hbar)}{[n]_q} g_{1,n}, \tag{6.3}$$

$$s_0 - \alpha_0^2 q^n \frac{[n-1]_q}{[n]_q}(\gamma(q)+\hbar)\cdot 1. \tag{6.4}$$

Meanwhile, the element s_- defined by formula (5.21) takes the form

$$s_- = \alpha_0 \frac{[n-1]_q+1}{[n]_q}((\gamma(q)+\hbar)-1)g_{1,n}. \tag{6.5}$$

Now let us introduce the algebra $\mathcal{A}_{\hbar,q}$ as the quotient of $T(\mathbf{g}_q)$ over the ideal generated by the elements listed in Proposition 6 (i), the elements (6.2)–(6.4) and all their descendants. Expressing μ via $\hbar$ and substituting it in formulae (5.9)–(5.11) we can realize the algebra $\mathcal{A}_{\hbar,q}$ as a subalgebra in End $V^q_\omega[[\hbar]]$ with a fixed $\omega(\alpha(\mu)$ is assumed to be expressed via the formula (6.1)).

This operator realization of the algebra $\mathcal{A}_{\hbar,q}$ implies that deformation $\mathcal{A} \to \mathcal{A}_{\hbar,q}$ is flat. In fact, it suffices to note that the algebra $\mathcal{A}_{\hbar,q}$ contains all components $V_{k(\omega_1+\omega_{n-1})}$, $k = 0, 1, 2, \ldots$. This follows from the fact that the images of the elements $g^k_{1,n}$ are non-trivial operators for any $k = 0, 1, 2, \ldots$ (recall that $|q| \neq 1$ and therefore q is not a root of unity).

The arguments analogous to the Nakayama lemma (cf. [2]) show that the algebra $\mathcal{A}_q = \mathcal{A}_{\hbar,q}/\hbar\mathcal{A}_{\hbar,q}$ also contains all components $V_{k(\omega_1+\omega_{n-1})}$ and therefore the deformation $\mathcal{A} \to \mathcal{A}_q$ is flat.

Our next aim is to verify that the Poisson pencil corresponding to the algebra $\mathcal{A}_{\hbar,q}$ is just that (1.3) with R-matrix (1.4). To do this it suffices to compute the brackets corresponding to one parameter deformations $\mathcal{A} \to \mathcal{A}_q$ and $\mathcal{A} \to \mathcal{A}_{\hbar,q}/(q-1)\mathcal{A}_{\hbar,q}$. It is easy to see that the algebra $\mathcal{A}_{\hbar,q}/(q-1)\mathcal{A}_{\hbar,q}$ is just that discussed in the beginning of this section. Therefore the corresponding Poisson bracket is proportional to the KKS one.

Consider now the algebra $\mathcal{A}_q$. Let us remark that this algebra differs from analogous one-parameter algebras from Donin and Shnider [11] and Donin and Gurevich [8]. The latter algebras were $\tilde{S}$-commutative where the operator $\tilde{S}$ is defined in Section 4 and the algebra $\mathcal{A}_q$ is no longer $\tilde{S}$-commutative. Instead of the relation $s_- = 0$ taking place in an $\tilde{S}$-commutative algebra we have now (6.5) with $\hbar = 0$. In [8] it has been shown that the Poisson bracket corresponding to an $\tilde{S}$-commutative algebra on a symmetric orbit is proportional to the R-matrix one.

The $\tilde{S}$-commutativity default of the algebra $\mathcal{A}_q$ is measured by the r.h.s. of formula (6.5). In the quasiclassical limit this term gives rise to a contribution proportional to the KKS bracket. This completes the proof.

In this connection the following question arises: what algebra of the family $\mathcal{A}_{\hbar,q}$ can be considered a q-analogue of the commutative algebra of functions on the $\mathbf{CP}^n$ type orbits and therefore can be called a *quantum or braided* $\mathbf{CP}^n$ type orbit? In [8] (following [11]) $\tilde{S}$-commutative algebras were considered in such a role.

However, from the representation theory point of view it is more reasonable to consider the algebra $\mathcal{A}_q$ as a "quantum (braided) orbit of $\mathbf{CP}^n$ type" since it is the only algebra from the family $\mathcal{A}_{\hbar,q}$ which cannot be represented in a q-deformed Verma module $V_{\mu\omega_1}$ with any μ. From this point of view it is a singular point like in the classical case.

One more way to define a version of a q-commutative algebra is discussed in [6] (cf. footnote 4).

So, there is no universal way to single out a braided analogue of a commutative algebra from the family $\mathcal{A}_{\hbar,q}$. All the above candidates for this role have their own motivations.

Let us remark that in $sl(2)$ case our approach leads to Podles' quantum sphere (cf. [32] where this algebra is also equipped with an involution $*$). We do not consider here the problem of a proper definition of involution operators (cf. [10] for a discussion on this problem). We could only emphasize that in our approach all representations of algebras in question are $U_q(\mathbf{g})$-morphisms. So, if we want to consider a $*$-representation theory of this algebra we must first introduce a $*$-operator in the space End V in the spirit of super-theory: the classical property $(AB)^* = B^*A^*$ will fail.

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